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SURFACE RADIATION INTEGRALS IN AERODYNAMIC SOUND

Final Report

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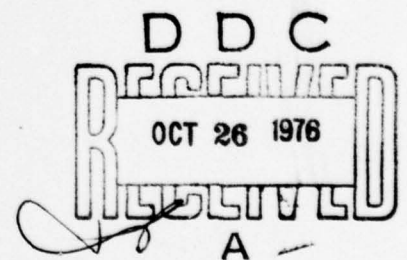MORRIS P. ISOM

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SURFACE RADIATION INTEGRALS IN AERODYNAMIC SOUND

I. INTRODUCTION

The sound field produced by a rigid surface in arbitrary motion was described in general terms by Ffowcs Williams and Hawkings in 1968 (Reference 1). Several alternative descriptions of the radiative component of the far pressure field were given. Each formula is an asymptotic solution to an inhomogeneous wave equation that is a generalization of the wave equation used by Lighthill to describe aerodynamic sound.

One of the formulations of Reference 1 was singled out by Farassat (References 2 and 3) as suitable for detailed noise calculations. His results are a considerable clarification of the original Ffowcs Williams and Hawkings analysis, and he has extended them to the point where, in principle, they are all that are necessary for numerical calculation of a sound field that includes non-compact source distributions.

The unusual and useful feature of Farassat's work is the coordinate space in which radiation integrals are expressed. This space is a temporal and spatial subspace of the complete four-dimensional space and time coordinate system. The possibly singular Doppler factor is suppressed in all radiation integrals in this coordinate system, and this result is an important advantage of the method. The elimination of an aerodynamic (Doppler) singularity from radiation integrals is accomplished at the expense of a geometric singularity that appears in all radiation integrands, and this new singularity occurs at all Mach numbers. The singularity appears when the retarded unit radiation vector from a point on the moving surface to the distant observer is parallel to the surface normal vector. Farassat

has shown in Reference 2 that the singularity is integrable for a useful class of surface geometries. It seems likely that it can be accomodated in numerical calculations, but whether it will be a minor or major annoyance in specific applications is not yet clear. In any case, the *methods and results* of References 2 and 3 may be considered as alternatives to Lowson's formulation of the aerodynamic sound problem (Reference 4) and to generalizations of Lowson's work that appear in Reference 1.

Here we intend to return to the Lowson description of the sound field in its generalized form stated by Ffowcs Williams and Hawkings. The motivation is based partly on the fact that the results of References 1 and 2 strongly suggest that the Doppler factor, which is important in low speed problems, appears to play no role in a high speed problem. It would therefore seem likely that it can be eliminated from the surface-fixed *coordinate description of the sound problem as originally given by* Lowson and generalized in Reference 1. This conjecture is true to an interesting and significant extent for the problem of sound radiated by a rotating and translating surface. However, our results have one possible disadvantage: while all radiation integrals have neither a Doppler or geometric singularity in them, they all appear with time derivatives in front of them. A considerable and successful effort was expended in References 1 and 2 to get all derivatives of radiation integrals under the integral sign. The formalism converts derivatives with respect to observer coordinates and time to derivatives with respect to coordinates defined on the moving surface. These transformations are essential if the main purpose is to obtain rough estimates of radiation integrals. However, they have the effect of increasing the strength of geometric singularities that must be integrated. On the other hand, the appearance of time derivatives before integrals with well-behaved integrands does not appear

to pose any problem for numerical calculations, and this form may in fact make numerical calculations easier.

General results will be given here for a particular class of radiation integrals. These are surface radiation integrals, or integrals of shell distributions of sources as they are called in Reference 1. They make an important and possibly dominant contribution to far field sound. The general formulation is intended for application to the problem of calculating the sound field from a helicopter rotor blade with transonic conditions in the blade tip region. The problem therefore includes non-compact source distributions on the blade surface and radiation from shock waves carried by the blade.

2. SURFACE INTEGRALS AND AN INTEGRAL THEOREM

Let S be the surface of a rigid body that rotates with constant angular velocity $\bar{\omega}$ about an axis fixed in space and that translates with forward velocity $\bar{v}$ in a plane perpendicular to $\bar{\omega}$. The contribution to far field pressure p from a surface distribution of sources of strength F on S is

$$4\pi p = \frac{\partial}{\partial t} \int \frac{F \cos\theta}{cr} \frac{dS}{1-M_r} \quad . \tag{2.1}$$

Notation is the same as in Reference 1; $\partial/\partial t$ is time differentiation in a reference frame fixed relative to undisturbed air and c is the constant sound speed of the undisturbed medium; θ is the angle between the unit radiation vector, directed from a point on S to the observer, and the outward surface normal vector; r is the distance from an integration point on S to the observer, and M_r is the component of the body surface Mach number in the radiation direction. Integration is

over the body surface S with the integrand evaluated at retarded time $\tau = t - r/c$.

Let $\bar{\eta} = (\eta_1, \eta_2, \eta_3)$ be a Cartesian system of coordinates fixed relative to S and let $\bar{x} = (x_1, x_2, x_3)$ be observer position in a frame fixed relative to undisturbed air. Then F, $\cos\theta$ and M_r are generally explicit functions of position $\bar{\eta}, \bar{x}$ and t through their dependence on retarded time τ, which is a function of $\bar{\eta}, \bar{x}$ and t. To emphasize this dependence, (2.1) is written as

$$4\pi\, p = \frac{\partial}{\partial t} \int F(\tau(\bar{\eta}, \bar{x}, t), \bar{\eta}) \, \frac{\cos\theta \; dS}{cr(1-M_r)} \quad , \tag{2.2}$$

with similar τ and $\bar{\eta}$ arguments for $\cos\theta$ and M_r.

The general representation of sound from a surface integral has a Doppler factor $|1-M_r|^{-1}$, and thus includes radiation Mach numbers M_r that may be supersonic. Here we assume that $M_r \leq 1$ and exclude supersonic surface motion. Of course, there may still be regions of supersonic flow over S and embedded shock waves.

The integral in (2.2) will be transformed in the surface fixed frame so that the factor $(1-M_r)^{-1}$ is suppressed. The transformation is based on some identities and an elementary theorem of differential geometry. We must first define the operations of total surface divergence and partial surface divergence relative to the surface S.

Let $\bar{F}$ be an arbitrary vector function of retarded time $\tau(\bar{\eta}, \bar{x}, t)$ and position $\bar{\eta}$ on S. Let $\bar{n}(\bar{\eta})$ be the unit outward normal vector on S specified in $\bar{\eta}$ coordinates, so that $\bar{n}$ is independent of τ. The radiation vector $\hat{r}(\tau)$ is the unit vector from a point on S to the observer. Two unit vectors $\bar{l}(\tau, \bar{\eta})$ and $\bar{m}(\tau, \bar{\eta})$, both tangent to S, are defined as in Reference 1 by

$$\bar{\Gamma} = \frac{\hat{r} \times \bar{n}}{|\hat{r} \times \bar{n}|} \quad , \quad \bar{m} = \bar{n} \times \bar{\Gamma} \quad , \quad |\hat{r} \times \bar{n}| = \sin\theta \geq 0 \quad , \tag{2.3}$$

so that $\bar{\Gamma}$, $\bar{m}$ and $\bar{n}$ form a right-handed triplet of unit vectors on S , specified in coordinates fixed to S . The vectors $\bar{\Gamma}$ and $\bar{m}$ play a fundamental role in Farassat's work. Here they may be considered as vectors derived from the more fundamental unit vectors $\hat{r}$ and $\bar{n}$ and always expressible in terms of $\hat{r}$ and $\bar{n}$.

We define the total surface divergence of any vector $\bar{F}$ defined on S by

$$\nabla_S \cdot \bar{F} = \bar{\Gamma} \cdot \nabla\bar{F} \cdot \bar{\Gamma} + \bar{m} \cdot \nabla\bar{F} \cdot \bar{m} \quad . \tag{2.4}$$

The gradient operator ∇ in (2.4) is with respect to $\bar{\eta}$ coordinates, and requires differentiation with respect to the implicit dependence of $\bar{F}$ on $\bar{\eta}$ through the dependence of $\bar{F}$ on $\tau(\bar{\eta})$, and also with respect to the explicit dependence of $\bar{F}$ on $\bar{\eta}$. The partial surface divergence of $\bar{F}$ is defined by

$$\nabla_{PS} \cdot \bar{F} = \bar{\Gamma} \cdot \nabla_p \bar{F} \cdot \bar{\Gamma} + \bar{m} \cdot \nabla_p \bar{F} \cdot \bar{m} \quad ,$$

where the partial gradient operator ∇_p is with respect to the explicit dependence of $\bar{F}$ on $\bar{\eta}$ only, with the $\tau(\bar{\eta})$ component of $\bar{F}$ held fixed. The total surface divergence of $\bar{F}$ defined in (2.4) can then be written

$$\nabla_S \cdot \bar{F} = \frac{\partial \bar{F}}{\partial \tau} \cdot \nabla_S \tau + \nabla_{PS} \cdot \bar{F} \quad .$$

The surface gradient of τ is defined by

$$\nabla_S \tau = \nabla\tau - (\bar{n} \cdot \nabla\tau)\,\bar{n} \quad ,$$

and is

$$\nabla_S \tau = \frac{\bar{m} \sin\theta}{c(1-M_r)} \quad .$$

Total surface divergence of $\bar{F}$ is therefore

$$\nabla_S \cdot \bar{F} = \frac{\partial \bar{F}}{\partial \tau} \cdot \frac{\bar{m} \sin\theta}{c(1-M_r)} + \nabla_{PS} \cdot \bar{F} \quad . \tag{2.5}$$

This identity plays a fundamental role in what follows.

There is a theorem in differential geometry that says that the integral over a closed surface S of the total surface divergence of a vector $\bar{F}$ everywhere tangent to S is zero (Weatherburn, Reference 5a):

$$\int \nabla_S \cdot \bar{F} \, dS = 0 \quad , \quad \bar{F} \text{ tangent to closed } S \quad . \tag{2.6}$$

If $\bar{F}$ is tangent to S, integration of (2.5) over S gives

$$\int \frac{\partial \bar{F}}{\partial \tau} \cdot \frac{\bar{m} \sin\theta}{c(1-M_r)} \, dS = - \int \nabla_{pS} \cdot \bar{F} \, ds \quad . \tag{2.7}$$

The theorem must be understood with the restriction that $\bar{F}$ is not too singular on S; this consideration plays a guiding role in what follows.

The integrand on the left-hand side of (2.7) has a Doppler factor, while that on the right-hand side is free of this factor. If $\bar{F}$ is a reasonably well behaved function on S, then $\nabla_{pS} \cdot \bar{F}$ can never produce singular behavior on S because the $\tau(\bar{\eta})$ component of $\bar{F}$ is held fixed in this operation. However, what is required is not the integrand that appears on the left of (2.7), but rather an integrand of the form $F \cos\theta/(1-M_r)$ that appears in the fundamental radiation integral of (2.2). The identity (2.7) must be further defined so that the basic radiation integral of (2.2) emerges in terms of less singular integrals.

3. SIMPLE IDENTITIES

Equation (2.5) can be rewritten identically as

$$\nabla_S \cdot \bar{F} = \frac{1}{c(1-M_r)} \frac{\partial}{\partial \tau}(\bar{m} \cdot \bar{F} \sin\theta) - \frac{\bar{F} \cdot}{c(1-M_r)} \frac{\partial}{\partial \tau}(\bar{m} \sin\theta) + \nabla_{PS} \cdot \bar{F} \quad . \quad (3.1)$$

The τ - derivative of $\bar{m} \sin\theta$ is

$$\frac{\partial}{\partial \tau}(\bar{m} \sin\theta) = \dot{\hat{r}} - (\bar{n} \cdot \dot{\hat{r}}) \bar{n} \quad . \quad (3.2)$$

The super dot means $\partial/\partial\tau$ at constant $\bar{\eta}$ and $\dot{\hat{r}} = -\bar{\omega} \times \dot{\hat{r}}$. For any function F , scalar or vector, we have the identity

$$\frac{\partial F}{\partial t} = \frac{\partial F}{\partial \tau} \frac{\partial \tau}{\partial t} = \frac{1}{1-M_r} \frac{\partial F}{\partial \tau} \quad . \quad (3.3)$$

Use of (3.2) and (3.3) in (3.1) gives

$$\bar{\nabla}_S \cdot \bar{F} = \frac{1}{c} \frac{\partial}{\partial t} (\bar{m} \cdot \bar{F} \sin\theta) - \frac{\bar{F}}{c(1-M_r)} \cdot [\dot{\hat{r}} - (\bar{n} \cdot \dot{\hat{r}})\bar{n}] + \nabla_{PS} \cdot \bar{F} \quad . \quad (3.4)$$

This identity is made useful in radiation integrals by defining the vector $\bar{F}$ by

$$\bar{F} = \bar{\lambda} (\tau(\bar{\eta}),\bar{\eta}) \, F \, (\tau(\bar{\eta}),\bar{\eta}) \quad ,$$

where now $\bar{\lambda}$ and F are an arbitrary vector and scalar defined on S . We write $\bar{\lambda}$ in component form as $\bar{\lambda} = \alpha \bar{\Gamma} + \beta \bar{m} + \gamma \bar{n}$. The $\bar{n}$ component of $\bar{\lambda}$ cancels throughout equation (3.4). We therefore redefine $\bar{\lambda}$ by $\bar{\lambda} = \bar{\alpha} \times \bar{n}$, where $\bar{\alpha}$ is arbitrary. Equation (3.4) takes the form

$$\nabla_S \cdot \bar{\alpha} \times \bar{n} \, F = \frac{1}{c} \frac{\partial}{\partial t}(F \, \bar{\alpha} \times \bar{m} \cdot \bar{n} \sin\theta) - \frac{F}{(c(1-M_r))} \bar{\alpha} \times \bar{n} \cdot \dot{\hat{r}} + \nabla_{PS} \cdot \bar{\alpha} \times \bar{n} \, F . \quad (3.5)$$

The vector $\bar{\alpha}$ can be written in component form as $\bar{\alpha} = A\bar{\Gamma} + B\bar{m}$, where A and B are arbitrary scalars defined on S. Equation (3.5) becomes

$$\nabla_S \cdot (B\bar{\Gamma} - A\bar{m})F = -\frac{\partial}{\partial t}\left(\frac{FA\sin\theta}{c}\right) + \frac{F}{c(1-M_r)}(A\bar{\omega}\cdot\bar{\Gamma}\cos\theta - B\bar{\omega}\cdot\bar{N})$$

$$+ \nabla_{PS}\cdot(B\bar{\Gamma} - A\bar{m})F \quad . \tag{3.6}$$

The unit vector $\bar{N} = \bar{\Gamma}\times\hat{r}$. Some simplification results if we write A and B as $A = a$, $B = -b\cos\theta$. The identity (3.6) is then

$$\nabla_S\cdot(b\bar{\Gamma}\cos\theta + a\bar{m})\,F = \frac{\partial}{\partial t}\left(\frac{Fa\sin\theta}{c}\right) - \frac{F\cos\theta}{c(1-M_r)}(a\bar{\omega}\cdot\bar{\Gamma} + b\bar{\omega}\cdot\bar{N})$$

$$+ \nabla_{PS}\cdot(b\cos\theta\,\bar{\Gamma} + a\bar{m})\,F \quad . \tag{3.7}$$

This equation is actually two equations determined by assuming $a \neq 0$, $b = 0$, and then $a = 0$, $b \neq 0$. We require the combined form (3.7). The identity (3.7) is used to solve for a term of the form $F\cos\theta/(1-M_r)^{-1}$. To do so, define a new scalar function G by

$$G = F(a\bar{\omega}\cdot\bar{\Gamma} + b\bar{\omega}\cdot\bar{N}) \quad .$$

This factor appears in the second term on the right-hand side of (3.7). In terms of arbitrary scalars a,b, and G, equation (3.7) is

$$\nabla_S\cdot\left[\frac{(b\bar{\Gamma}\cos\theta + a\bar{m})\,G}{a\bar{\omega}\cdot\bar{\Gamma} + b\bar{\omega}\cdot\bar{N}}\right] = \frac{\partial}{\partial t}\left[\frac{Ga\sin\theta}{c(a\bar{\omega}\cdot\bar{\Gamma} + b\bar{\omega}\cdot\bar{N})}\right] - \frac{G\cos\theta}{c(1-M_r)}$$

$$+ \nabla_{PS}\cdot\left[\frac{(b\bar{\Gamma}\cos\theta + a\bar{m})G}{a\bar{\omega}\cdot\bar{\Gamma} + b\bar{\omega}\cdot\bar{N})}\right] \tag{3.8}$$

Integration of (3.8) over S and application of the integral theorem (2.6) gives the identity

$$\int\frac{G\cos\theta}{(1-M_r)}\,dS = \frac{\partial}{\partial t}\int\frac{Ga\sin\theta}{a\bar{\omega}\cdot\bar{\Gamma} + b\bar{\omega}\cdot\bar{N}}\,dS$$

$$+ c\int\nabla_{PS}\cdot\left[\frac{(b\bar{\Gamma}\cos\theta + a\bar{m})G}{a\bar{\omega}\cdot\bar{\Gamma} + b\bar{\omega}\cdot\bar{N}}\right]\,dS \tag{3.9}$$

The scalars a and b in equation (3.9) are still arbitrary and
at our disposal. It is evident that a variety of identities can be generated
by various choices of these scalar functions. One identity that is
particularly useful is based on setting

$$a = \bar{\omega} \cdot \bar{\Gamma} = \omega_T \quad , \quad b = \bar{\omega} \cdot \bar{N} = \omega_N \quad .$$

The denominators in (3.9) are then

$$a\omega_T + b\omega_N = (\bar{\omega} \cdot \bar{\Gamma})^2 + (\bar{\omega} \cdot \bar{N})^2 \quad . \tag{3.10}$$

The unit vectors $\bar{\Gamma}$, $\hat{r}$ and $\bar{N} = \bar{\Gamma} \times \hat{r}$ are an orthogonal system
defined at each point of S . The angular velocity $\bar{\omega}$ can therefore
be written as

$$\bar{\omega} = (\bar{\omega} \cdot \bar{\Gamma}) \, \bar{\Gamma} + (\bar{\omega} \cdot \hat{r})\hat{r} + (\bar{\omega} \cdot \bar{N})\bar{N} \quad .$$

Then (3.10) is

$$a\omega_T + b\omega_N = \omega^2 - (\bar{\omega} \cdot \hat{r})^2 \quad .$$

Let us call this quantity

$$\Omega^2 = \omega^2 - (\bar{\omega} \cdot \hat{r})^2 \quad . \tag{3.11}$$

The identity (3.9) becomes

$$\int \frac{G \cos\theta}{(1-M_r)} \, dS = \frac{\partial}{\partial t} \int \Omega^{-2} \, G \, \omega_T \, \sin\theta \, dS + c \int \nabla_{PS} \cdot \left[\Omega^{-2} (\omega_N \bar{\Gamma} \cos\theta + \omega_T \bar{m}) G \right] dS \tag{3.12}$$

Now $\bar{\eta}$ - derivatives of $\hat{r}$ with τ held constant are all $O(1/r)$, and
therefore so are $\bar{\eta}$ - derivatives of Ω with τ constant. Furthermore,
the τ derivative of Ω is identically zero. Hence Ω^{-2} is either
asymptotically or identically zero with respect to all differential
operations in equation (3.12), and this factor may be treated as a

constant factor in that equation. Thus we finally have the identity

$$\int \frac{G \cos\theta}{1-M_r} \, dS = \Omega^{-2} \frac{\partial}{\partial t} \int G \, \bar{\omega} \times \hat{r} \cdot \bar{n} \, dS + c\Omega^{-2} \int \nabla_{PS} \cdot \left[(\omega_N \bar{\Gamma} \cos\theta + \omega_T \bar{m}) G \right] dS \ .$$
(3.13)

The relation $\omega_T \sin\theta = \bar{\omega} \times \hat{r} \cdot \bar{n}$ has been used for the second integral of (3.13). The left-hand side of this equation has the form required in the radiation integral (2.2). However, (3.13) can be further simplified in a useful way. The vector in the last integrand of (3.13) can be written as

$$\omega_N \bar{\Gamma} \cos\theta + \omega_T \bar{m} = - \bar{\omega} \times \bar{n} + (\bar{\omega} \cdot \hat{r}) \hat{r} \times \bar{n} \ .$$
(3.14)

This identity is easily verified by taking the scalar product of both sides successively with $\bar{\Gamma}$ and $\bar{m}$. The last integrand of (3.13) is

$$\nabla_{PS} \cdot \left[(\omega_N \bar{\Gamma} \cos\theta + \omega_T \bar{m}) G \right] = (\omega_N \bar{\Gamma} \cos\theta + \omega_T \bar{m}) \cdot \nabla_{PS} G$$

$$+ G \nabla_{PS} \cdot (\omega_N \bar{\Gamma} \cos\theta + \omega_T \bar{m})$$

The last divergence in this equation is, using (3.14),

$$\nabla_{PS} \cdot (\omega_N \bar{\Gamma} \cos\theta + \omega_T \bar{m}) = - \nabla_{PS} \cdot (\bar{\omega} \times \bar{n}) + \nabla_{PS} \cdot \left[(\bar{\omega} \cdot \hat{r}) \hat{r} \times \bar{n} \right].$$
(3.15)

The angular velocity $\bar{\omega}$ is constant in (3.15), and the $\bar{\eta}$ - derivatives of the factor $\bar{\omega} \cdot \hat{r}$ are $O(1/r)$ because $\bar{\eta}$ - derivatives with τ constant of $\hat{r}$ are $O(1/r)$. Hence equation (3.15) is

$$\nabla_{PS} \cdot (\omega_N \bar{\Gamma} \cos\theta + \omega_T \bar{m}) = \bar{\omega} \cdot \nabla_{PS} \times \bar{n} - (\bar{\omega} \cdot \hat{r}) \hat{r} \cdot \nabla_{PS} \times \bar{n} \ .$$
(3.16)

But the <u>surface</u> curl of the unit normal surface vector $\bar{n}$ is zero; this is a fundamental relation of differential geometry shown, for example,

in Reference 5a. Hence equation (3.16) is

$$\nabla_{PS} \cdot (\omega_N \hat{r} \cos\theta + \omega_T \bar{m}) = 0 \quad,$$

and the third integrand of (3.13) is

$$\nabla_{PS} \cdot \left[(\omega_N \hat{r} \cos\theta + \omega_T \bar{m}) G \right] = -\bar{\omega} \times \bar{n} \cdot \nabla_{PS} G + (\bar{\omega} \cdot \hat{r}) \hat{r} \times \bar{n} \cdot \nabla_{PS} G \quad.$$

The fundamental identity (3.13) is therefore

$$\int \frac{G \cos\theta}{1-M_r} dS = \Omega^{-2} \frac{\partial}{\partial t} \int G \, \bar{\omega} \times \hat{r} \cdot \bar{n} \, dS$$

$$- c \, \Omega^{-2} \int \bar{\omega} \times \bar{n} \cdot \nabla_{PS} G \, dS + (\bar{\omega} \cdot \hat{r}) \Omega^{-2} \int \hat{r} \times \bar{n} \cdot \nabla_{PS} G \, dS \quad (3.17)$$

We now identify the function G in (3.17) with the source strength F/r in the radiation integral of equation (2.2). The contribution to far field pressure from sources distributed on S is

$$4\pi p = \frac{\Omega^{-2}}{c} \frac{\partial^2}{\partial t^2} \int F \, \bar{\omega} \times \hat{r} \cdot \bar{n} \, \frac{dS}{r} - \Omega^{-2} \frac{\partial}{\partial t} \int \bar{\omega} \times \bar{n} \cdot \nabla_{PS} F \, \frac{dS}{r}$$

$$+ \Omega^{-2} (\bar{\omega} \cdot \hat{r}) \frac{\partial}{\partial t} \int \hat{r} \times \bar{n} \cdot \nabla_{PS} F \, \frac{dS}{r} \quad. \quad (3.18)$$

If the observer is in the plane of rotation, $\bar{\omega} \cdot \hat{r} = 0$ and the last integral in (3.18) is zero. If the surface S is hovering (translational velocity $\bar{v} = 0$), the source strength F is independent of τ in coordinates fixed to S. The integrand of the second integral in (3.18) is therefore independent of τ and hence independent of t for hover, and the time derivative of this integral is zero. Therefore for hover and observer in the plane of rotation

$$4\pi p = \frac{\omega^{-2}}{c} \frac{\partial^2}{\partial t^2} \int F \, \bar{\omega} \times \hat{r} \cdot \bar{n} \, \frac{dS}{r} \quad. \quad (3.19)$$

The integral depends on observer position and time through the dependence of $\hat{r}$ on $\tau(\bar{\eta}, \bar{x}, t)$. Equation (3.19) appears to be a remarkably simple expression for this important case.[*]

4. GEODESIC PROPERTIES OF THE ROTATING SURFACE

The last integral in equation (3.18) is

$$\int \hat{r} \times \bar{n} \cdot \nabla_{PS} F \frac{dS}{r} = \int \sin\theta \, \bar{\Gamma} \cdot \nabla_{PS} F \frac{dS}{r} \quad . \tag{4.1}$$

This result follows from the definition of the unit vector $\bar{\Gamma}$ in equation (2.3). For any scalar F,

$$\bar{\Gamma} \cdot \nabla_S F = \frac{\partial F}{\partial \tau} \frac{\bar{\Gamma} \cdot \bar{m} \sin\theta}{c(1-M_r)} + \bar{\Gamma} \cdot \nabla_{PS} F \quad . \tag{4.2}$$

Since $\bar{\Gamma} \cdot \bar{m} = 0$, we find $\bar{\Gamma} \cdot \nabla_S F = \bar{\Gamma} \cdot \nabla_{PS} F$. Equation (4.1) is therefore

$$\int \hat{r} \times \bar{n} \cdot \nabla_{PS} F \frac{dS}{r} = \int \sin\theta \, \bar{\Gamma} \cdot \nabla_S F \frac{dS}{r} \quad . \tag{4.3}$$

For the second integrand in (4.3), we have

$$\sin\theta \, \bar{\Gamma} \cdot \nabla_S F \equiv \nabla_S \cdot (\bar{\Gamma} F \sin\theta) - F \nabla_S \cdot (\bar{\Gamma} \sin\theta) \quad .$$

The factor r^{-1} can always be treated as constant in derivatives with respect to $\bar{\eta}$ - coordinates and τ. Therefore (4.3) is

$$\int \hat{r} \times \bar{n} \cdot \nabla_{PS} F \frac{dS}{r} = \int \nabla_S \cdot (\frac{\bar{\Gamma} F \sin\theta}{r}) \, dS - \int F \nabla_S \cdot (\bar{\Gamma} \sin\theta) \frac{dS}{r} \quad . \tag{4.4}$$

The second integral is zero by the surface integral theorem (2.6). In the third integral of (4.4),

$$\nabla_S \cdot \bar{\Gamma} \sin\theta = \sin\theta \, \nabla_S \cdot \bar{\Gamma} + \cos\theta \, \bar{\Gamma} \cdot \nabla_S \theta \quad . \tag{4.5}$$

Now the unit vectors $\bar{\Gamma}$ and $\bar{m}$ form an orthogonal net on S. The total surface divergence of $\bar{\Gamma}$ is $\nabla_S \cdot \bar{\Gamma} = \gamma$, where γ is the

[*] But see the note added in proof, page 21.

geodesic curvature of the coordinate curve on S with $\bar{m}$ as unit tangent vector. The directional derivative $\bar{\Gamma} \cdot \nabla_S \theta$ in (4.5) is $\bar{\Gamma} \cdot \nabla_S \theta = \bar{\Gamma} \cdot \nabla_{PS}\theta$, and this equality results from (4.2) and $\bar{\Gamma} \cdot \bar{m} = 0$. Since $\bar{n} \cdot \hat{r} = \cos\theta$,

$$- \sin\theta \ \bar{\Gamma} \cdot \nabla_{PS}\theta = \bar{\Gamma} \cdot \nabla_{PS} \ \bar{n} \cdot \hat{r} = \frac{\partial}{\partial \bar{\Gamma}} \ \bar{n} \cdot \hat{r} \ , \tag{4.6}$$

where $\partial/\partial\bar{\Gamma}$ is differentiation along a $\bar{\Gamma}$ - curve with τ constant. The last term in this equation is

$$\frac{\partial}{\partial\bar{\Gamma}} \ \bar{n} \cdot \hat{r} = \hat{r} \cdot \frac{\partial\bar{n}}{\partial\bar{\Gamma}} = - \hat{r} \cdot (\varkappa_n \bar{\Gamma} + T\bar{m}) \ , \tag{4.7}$$

where T is the torsion of a geodesic curve tangent to a $\bar{\Gamma}$ - curve, and $\varkappa_n$ is normal curvature of an $\bar{m}$ - curve. The last equality in (4.7) follows from differential properties of the unit normal vector $\bar{n}$ as discussed in Chapter I of Volume II of Weatherburn's texts. Orthogonality of $\hat{r}$ and $\bar{\Gamma}$ reduces (4.7) to

$$\frac{\partial}{\partial\bar{\Gamma}} \ \bar{n} \cdot \hat{r} = - T \sin\theta \ .$$

Use of this relation in (4.6) gives

$$\bar{\Gamma} \cdot \nabla_{PS} \ \theta = T \ ,$$

and hence equation (4.5) is

$$\nabla_S \cdot \Gamma \sin\theta = \gamma \sin\theta + T \cos\theta \ .$$

The left-hand side of (4.4) can therefore be written

$$\int \hat{r} \times \bar{n} \cdot \nabla_{PS} F \ \frac{dS}{r} = - \int (\gamma \sin\theta + T \cos\theta) F \ \frac{dS}{r} \ . \tag{4.8}$$

Surface derivatives of F in the first integral of (4.8) have been eliminated. This result is accomplished at the expense of introducing the relatively unfamiliar geodesic properties of S .

Use of equation (4.8) in (3.18) gives an alternative expression for acoustic pressure due to surface integrals:

$$4\pi p = \frac{\Omega^{-2}}{c}\frac{\partial^2}{\partial t^2}\int F\,\bar{\omega}\times\hat{r}\cdot\bar{n}\,\frac{dS}{r} - \Omega^{-2}\frac{\partial}{\partial t}\int\bar{\omega}\times\bar{n}\cdot\nabla_{PS}F\,\frac{dS}{r}$$

$$- \Omega^{-2}(\bar{\omega}\cdot\hat{r})\frac{\partial}{\partial t}\int(\gamma\,\sin\theta + T\,\cos\theta)F\,\frac{dS}{r}\ . \tag{4.9}$$

Throughout the preceeding formulas, the factor Ω^2 is zero when the retarded radiation vector $\hat{r}$ is parallel to the angular velocity $\bar{\omega}$. This occurs when $\hat{r}$ is perpendicular to the translational velocity $\bar{V}$ and the angular velocity $\bar{\omega}\times\hat{r}$. Retarded times τ over which Ω^2 is zero or small therefore correspond to values of τ over which the radiation Mach number M_r is zero or small. This situation occurs when the lateral distance of the observer from the center of rotation is held fixed and the vertical distance becomes infinite in the limit $r\to\infty$. This observer position corresponds to the rotary surface being straight overhead (or below) at some retarded time. In this case, acoustic pressure can be determined by radiation integrals in which M_r is zero or small, and the generalized Lowson formulas are well suited to that kind of calculation. Equations (3.18) and (4.9) are useful for those observer positions at which the amplitude of radiated sound may be large. These positions may be in the plane of rotation, or on any line through the center of rotation that is inclined at an angle of less than 90° to the plane of rotation.

5. SHOCKS

1. The hovering rotor.

The shock wave on a hovering surface S rotating at constant angular velocity is stationary relative to S. The geometry of the shock

is independent of time, and the velocity relative to an inertial frame of points on the shock is solenoidal. The hovering shock wave may therefore be considered as a rigid surface. This description of the shock surface is no longer strictly true if the body surface is rotating and also advancing.

Let S' be the shock surface on a hovering rotor. Let $\bar{n}(\bar{\eta})$ be the unit vector normal to S' that points in the direction upstream of S', and let θ be the angle between $\bar{n}$ and the unit radiation vector $\hat{r}(\tau)$. The contribution to acoustic pressure from the shock is

$$4\pi cp = \frac{\partial}{\partial t} \int \Delta T_{rr} \cos\theta \frac{dS'}{r(1-M_r)} \quad . \tag{5.1}$$

Integration is over the shock surface and the integrand is evaluated at retarded time. The source strength ΔT_{rr} is the jump in the resolved Lighthill stress tensor T_{rr} across the shock. If the superscript (2) denotes the upstream side of S' and (1) the downstream side, then $\Delta T_{rr} = T_{rr}^{(2)} - T_{rr}^{(1)}$, and $T_{rr} = \hat{r}_i \hat{r}_j T_{ij}$.

Equation (5.1) is the equivalent, in surface fixed coordinates, of a similar contribution found by Farassat in his coordinate representation. However, he finds an additional term that involves the derivative in the radiation direction of ΔT_{rr}. That term appears to be in error and is omitted here.

The Doppler factor in (5.1) can be suppressed by transforming the integral along the lines used in sections 2 and 3, only now the shock surface S' is not closed and a more general version of the surface divergence theorem of equation (2.6) is required.

Let C be the closed curve that bounds the shock surface S'. Let $\bar{\ell}$ be a unit vector normal to C and tangent to S' and pointing away from S'. Let $\bar{u}$ be a unit vector tangent to C so that $\bar{\ell}, \bar{u}$ and $\bar{n}$ from a right handed system of unit vectors on C. If $\bar{F}(\tau(\bar{\eta}), \bar{\eta})$ is

any vector everywhere tangent to S', then (Weatherburn, Vol. 1)

$$\int \nabla_S \cdot \bar{F} \, dS' = \int \bar{F} \cdot \bar{\ell} \, ds \quad . \tag{5.2}$$

We again introduce unit vectors $\bar{\Gamma}$ and $\bar{m}$ tangent to S' as defined by equations (2.3), with total surface divergence $\nabla_S \cdot \bar{F}$ defined by equation (2.4). The contour integral on the right-hand side of (5.2) is taken around C in the positive direction determined by the tangent vector $\bar{u}$; ds is arc length along C.

Transformation of (5.1) now follows the same steps used to reduce integrals over closed surfaces, developed in sections 2 and 3. The basic surface divergence identity for an arbitrary scalar G is equation (3.8), which can be written as

$$c \, \Omega^{-2} \nabla_S \cdot [(\omega_N \bar{\Gamma} \cos\theta + \omega_T \bar{m}) G] = \Omega^{-2} \frac{\partial}{\partial t} (G \, \omega_T \sin\theta)$$
$$- \frac{G \cos\theta}{1-M_r} + c \, \Omega^{-2} \nabla_{PS} \cdot [(\omega_N \bar{\Gamma} \cos\theta + \omega_T \bar{m}) G] \, , \tag{3.8'}$$

with $\Omega^2 = \omega^2 - (\bar{\omega}\cdot\hat{r})^2$. The vector in parantheses in (3.8') is again

$$\omega_N \bar{\Gamma} \cos\theta + \omega_T \bar{m} = - \bar{\omega}\times\bar{n} + (\bar{\omega}\cdot\hat{r})\hat{r}\times\bar{n} \, . \tag{3.14'}$$

The partial surface divergence in (3.8') is again

$$\nabla_{PS} \cdot [(\omega_N \bar{\Gamma} \cos\theta + \omega_T \bar{m}) G] = (\omega_N \bar{\Gamma} \cos\theta + \omega_T \bar{m}) \cdot \nabla_{PS} G$$
$$= - \bar{\omega}\times\bar{n}\cdot\nabla_{PS} G + (\bar{\omega}\cdot\hat{r})\hat{r}\times\bar{n}\cdot\nabla_{PS} G \, , \tag{5.3}$$

using (3.14') in the last equality. Now substitute (3.14') and (5.3) in (3.8'), integrate over S' and apply the open surface integral theorem, equation (5.2). Identify the arbitrary function G in (3.8') with the shock source strength $\nabla T_{rr}/r$ in the radiation integral of equation (5.1). Note that the shock surface S' generally intersects the body surface S in a curve C_I that is part of the shock boundary C. The part of the boundary C that does not lie along the body surface is just the sonic line of the shock. On this part of C, $\Delta T_{rr} = 0$. The closed countour integral in (5.2) therefore reduces to an integral along the

intersection of the shock S' with the body S, and this is a line integral along C_1. The contribution to acoustic pressure from a shock is therefore

$$
4\pi \Omega^2 p = \frac{\partial^2}{\partial t^2} \int \bar{\omega} \times \hat{r} \cdot \bar{n} \, \Delta T_{rr} \frac{dS'}{cr}
$$

$$
- \frac{\partial}{\partial t} \int \bar{\omega} \times \bar{n} \cdot \nabla_{PS} \Delta T_{rr} \frac{dS'}{r}
$$

$$
+ (\bar{\omega} \cdot \hat{r}) \frac{\partial}{\partial t} \int \hat{r} \times \bar{n} \cdot \nabla_{PS} \Delta T_{rr} \frac{dS'}{r}
$$

$$
+ \frac{\partial}{\partial t} \int_{C_1} \bar{\omega} \times \bar{n} \cdot \bar{\ell} \, \Delta T_{rr} \frac{ds}{r}
$$

$$
- (\bar{\omega} \cdot \hat{r}) \frac{\partial}{\partial t} \int_{C_1} \hat{r} \times \bar{n} \cdot \bar{\ell} \, \Delta T_{rr} \frac{ds}{r} \tag{5.4}
$$

For a hovering rotor, the second and fourth integrals on the right-hand side of (5.4) are independent of t because their integrands are independent of retarded time τ. In this case, time derivatives of these integrals are zero. If the observer is in the plane of rotation, $\bar{\omega} \cdot \hat{r} = 0$. Therefore, for hover and an in-plane observer, the shock contribution to radiative pressure is

$$
4\pi p = \omega^{-2} \frac{\partial^2}{\partial t^2} \int \bar{\omega} \times \hat{r} \cdot \bar{n} \, \Delta T_{rr} \frac{dS'}{cr} \quad . \tag{5.5}
$$

This formula is analogous to equation (3.19) that gives the contribution to acoustic pressure from a closed hovering surface and an in-plane observer.[*]

2. Advancing rotor.

If the forward velocity of the rotor is not zero, the shock surface is not necessarily stationary relative to the body surface. In addition, the shape of shock surface may change as it grows and collapses with the passage of the blade tip through the advancing azimuthal region. The deformation of the shock means that the velocity of points on it, relative to an inertial frame, is no longer solenoidal. Throughout

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[*]But see the note added in proof, page 21.

Reference 3 it was assumed that both the body and shock surfaces are rigid, and hence that their velocities have zero divergence. It may be that a changing shock geometry produces only minor variations in far field pressure, but the assumption of a rigid shock geometry is probably not quite correct, and possible errors introduced by this assumption are somewhat obscure. The results of Reference 3 could be altered to account for unsteady shock geometry; the changed analysis would begin with equation (5) of that paper and would apply only to contributions from the jump in the Lighthill stress tensor across the shock. It is also assumed in Reference 3 that the sign of the local first curvature of the shock does not change from point to point.

Expressions for the contribution to acoustic pressure from an advancing shock are greatly simplified if the shock can be considered as rigid, and this assumption will be made here. The boundary C of the shock surface S' must, of course, be considered as a function of time. We shall also assume that the acceleration of the shock surface relative to the body is small, or that variations in this relative velocity occur over such a small time inteval that they may be neglected. This assumption means that the τ-derivative of the unit radiation vector $\hat{r}$ is still $\dot{\hat{r}} = -\bar{\omega} \times \hat{r}$ and the factor Ω^2 defined by equation (3.11) can be regarded as independent of τ and $\hat{\eta}$ coordinates. Assumptions about shock geometry and acceleration can be removed at the expense of added complication. The extent to which these assumptions are reasonable may be clarified when we have detailed calculations that show shock behavior on advancing blades.

With the above assumptions, equation (5.4) gives the contribution to acoustic pressure from the unsteady advancing shock wave. The second integral in equation (5.4) can be rewritten using equation (4.8). Surface

derivatives of ΔT_{rr} are thereby eliminated from the integral, and far field pressure will depend on local geodesic properties of the shock surface.

5. SUMMARY AND CONCLUSIONS

Equation (3.18) gives the contribution to acoustic pressure from a distribution of sources of strength F on the rotating and translating surface; the corresponding contribution from shock waves is equation (5.4). If the surface is hovering and the observer is in the plane of rotation, these formulas simplify to their respective counterparts, equations (3.19) and (5.5). Each radiation integral is free of singularities as long as the surface advancing tip Mach number is not supersonic. The integrals are a function of aerodynamic data on the blade and shock surfaces, and this information is assumed to be known from near field calculations. The near field numerical calculations are always carried out in a coordinate system fixed to the accelerating surface. The required data can therefore be inserted in radiation integrals with a minimum of transformation.

The source strength F in equations (3.18) and (3.19) is generally a linear combination of the Lighthill stress tensor (resolved twice in the radiation direction), the pressure distribution on the blade surface and also the surface radiation Mach number. The complete far field pressure is composed of body surface and shock contributions according to equations (3.18) and (5.4); in addition, it depends on the derivative in the radiation direction of the Lighthill stress tensor, which must be integrated over the volume exterior to the body and shock surfaces. This residual volume integral is not discussed here. Its integrand is a second order term, and the integral may be negligible.

7. REFERENCES

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NOTE ADDED IN PROOF

The contribution to acoustic pressure from sources distributed on a hovering surface is given by equation (3.19); the contribution from the shock on the hovering surface is equation (5.5). Both of these results are based on the assumption that the source strengths are independent of time, and therefore independent of retarded time, in coordinates fixed to the surface. This assumption is false, or rather it is half true, for both the body surface and the shock. The body surface source strength F in equation (3.19) generally depends on surface pressure and density, and both of these quantities are independent of retarded time in $\bar{\eta}$ coordinates. However, F also depends on the component of the disturbance velocity in the radiation direction, and this quantity is a function of retarded time τ through the dependence of the radiation vector $\hat{r}$ on τ. Thus F has components for which equation (3.19) is correct, and also components for which the full equation (3.18) is required. The same must be said of the shock source strength ΔT_{rr}: it has components that depend on the jumps in pressure and density across the shock, and both of these quantities are independent of τ in $\bar{\eta}$ coordinates. Equation (5.5) is correct for the contribution from these discontinuities. However, ΔT_{rr} also has components dependent on the disturbance velocity resolved in the radiation direction $\hat{r}$, and these contributions to far field pressure must be expressed by the full equation (5.4).